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The CP(n) Model on Noncommutative Plane

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We construct the consistent $CP(n)$ model on noncommutative plane. The Bogomolny bound on the energy is saturated by (anti-)self-dual solitons with integer topological charge, which is independent of their scaling and orientation. This integer quantization is satisfied for our general solutions, which turns out regular everywhere. We discuss the possible implication of our result to the instanton physics in Yang-Mills theories on noncommutative $\mathbf{R}^4$.

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1 Introduction

Quantum field theory on a noncommutative space has been proved to be useful in understanding various physical phenomena, like as various limits of M-theory compactification [1, 2], low energy effective field theory of D-branes with constant Neveu-Schwarz B-field background [3, 4], and quantum Hall effect [5]. Although noncommutative field theories are non-local, they appear to be highly constrained deformation of local field theory. Thus it may help understanding non-locality at short distances in quantum gravity.

Noncommutative field theory means that fields are thought of as functions over noncommutative spaces. At the algebraic level, the fields become operators acting on a Hilbert space as a representation space of the noncommutative space. Since the noncommutative space resembles a quantum phase space (with noncommutativity θ playing the role of $\hbar$), it exhibits an interesting spacetime uncertainty relation, which causes a UV/IR mixing [6] and a teleological behavior [7]. Also, for nonzero θ, there can be nonperturbative effects in the form of soliton solutions even at the classical level and it could not have a smooth limit. Indeed, several such solutions have recently appeared [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18].

In this Letter we will show the existence of nonperturbative solutions in the $CP(n)$ model on noncommutative two plane. The $CP(n)$ model, even though it consists only of scalar fields, enjoys local gauge invariance and exhibits many similarities to instantons in four-dimensional Yang-Mills theory, such as the existence of self-dual soliton solution with scale and orientation parameters [19, 20]. In addition, it has many applications to condensed matter systems [21]. However, since the $CP(n)$ model is relatively simpler than four-dimensional noncommutative Yang-Mills theory, it will be very useful "toy model" to investigate various questions of noncommutative gauge theory if the ordinary $CP(n)$ model can be generalized to noncommutative space. We will demonstrate here it is the case.

In Section 2, we construct the consistent $CP(n)$ model on noncommutative plane. In Section 3, the Bogomolny bound on the energy is considered and it is shown that it is saturated by (anti-)self-dual solutions. In Section 4, we solve the soliton solutions for the (anti-)self-dual equations and show that their topological charges, which are independent of their scaling and orientation, are quantized. However, we point out that this integer quantization is satisfied only for the field configurations without any singularity in the commutative sense. Also we argue that our solution is the most general BPS solution. In Section 5, we summarize our results with a brief discussion

of the possible implication to the instanton physics in Yang-Mills theories on noncommutative $\mathbf{R}^4$ [9], including the problem for noncommutative instanton solutions discussed in [14].

2 CP(n) Model

We consider the (2+1)-dimensional field theory whose space is noncommutative two plane. The coordinates x, y of this noncommutative plane satisfy the relation

$$[x, y] = i\theta \tag{1}$$

with $\theta > 0$. This noncommutative plane has not only the translation symmetry but also rotational symmetry. One can see that the parity operation $(x, y) \to (x, -y)$ is broken on noncommutative plane. The classical field on this noncommutative space is an element $\Phi(t, x, y)$ in the algebra $\mathcal{A}_\theta$ defined by x, y with the relation (1).

Introduce the complex coordinates

$$z = \frac{x + iy}{\sqrt{2}}, \quad \bar{z} = \frac{x - iy}{\sqrt{2}}, \tag{2}$$

which satisfy

$$[z, \bar{z}] = \theta > 0. \tag{3}$$

This commutation relation is that of the creation and annihilation operators for a simple harmonic oscillator and so one may use the simple harmonic oscillator Hilbert space $\mathcal{H}$ as a representation of (1). The ground state is $|0>$ such that $z|0>= 0$ and $|n>= \bar{z}^n/\sqrt{\theta^n n!}|0>$ so that

$$z|n>= \sqrt{\theta n}|n-1>, \quad \bar{z}|n>= \sqrt{\theta(n+1)}|n+1> . \tag{4}$$

The integration over noncommutative two plane becomes the trace over its Hilbert space, which respect the translation symmetry:

$$\int d^2x \mathcal{O} \to \mathrm{Tr}\mathcal{O} = 2\pi\theta \sum_{n\geq 0} < n|\mathcal{O}|n > . \tag{5}$$

The $CP(n)$ manifold is defined by an $(n{+}1)$-dimensional complex vector $\Phi = (\phi_1, \phi_2, ..., \phi_{n+1})$ of unit length with the equivalence relation under the overall phase rotation $\Phi \sim e^{i\alpha}\Phi$ [19, 20]. This complex projective space of real dimension $2n$ is equivalent of the coset space $U(n+1)/U(1) \times U(n)$. (It is quite straightforward to generalize our consideration here to the general Grassmanian models with the manifold $G(n, m) = U(n)/U(m) \times U(n-m)$ [22].)

Here we are interested in the $CP(n)$ model on the noncommutative plane. Thus the field variable $\Phi(t,x,y)$ becomes an operator acting on $\mathcal{H}$. The spatial derivatives are

$$\partial_x\Phi = i\theta^{-1}[y,\Phi], \quad \partial_y\Phi = -i\theta^{-1}[x,\Phi]. \tag{6}$$

The natural Lagrangian for the $CP(n)$ model turns out to be

$$L_1 = \mathrm{Tr}\left(\partial_\mu\Phi^\dagger\partial^\mu\Phi + (\Phi^\dagger\partial_\mu\Phi)(\Phi^\dagger\partial^\mu\Phi)\right) \tag{7}$$

with the constraint

$$\Phi^\dagger\Phi - 1 = 0. \tag{8}$$

This theory has a global $U(n+1)$ symmetry and a local $U(1)$ symmetry

$$\Phi(x) \to \Phi(x)g(x) \tag{9}$$

which removes the degrees of freedom for the overall phase of Φ. The $U(1)$ gauge transformation acts on the right side, which leaves the constraint (8) invariant. This ordering of the gauge transformation is the key point which makes the whole model work.

This Lagrangian with the constraint (8) can be rewritten as

$$L_2 = \mathrm{Tr}\left(D_\mu\Phi^\dagger D^\mu\Phi + \lambda(\Phi^\dagger\Phi - 1)\right) \tag{10}$$

with

$$D_\mu\Phi = \partial_\mu\Phi - i\Phi A_\mu, \tag{11}$$

where $A_\mu(x)$ is the $U(1)$ gauge field without its kinetic term and $\lambda(x)$ is a Lagrangian multiplier to incorporate the constraint (8). This Lagrangian is invariant under the local gauge transformation defined by (9) and

$$A_\mu(x) \to g^\dagger A_\mu g - ig^\dagger\partial_\mu g. \tag{12}$$

As there is no gauge kinetic term, one can solve the A_μ equation to get

$$A_\mu = -i\Phi^\dagger\partial_\mu\Phi \tag{13}$$

which shows that the gauge transformations (9) and (12) are consistent with one another. After using Eqs. (8) and (13), the second Lagrangian (10) becomes the first Lagrangian (7), as it should be.

Note that $\Phi^\dagger D_\mu \Phi = 0$ and the field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \tag{14}$$

$$= -i(D_\mu \Phi^\dagger D_\nu \Phi - D_\nu \Phi^\dagger D_\mu \Phi) \tag{15}$$

which is the curvature tensor $[D_\mu, D_\nu]\Phi = -i\Phi F_{\mu\nu}$. From the field equation for Φ,

$$D_\mu D^\mu \Phi - \Phi\lambda = 0, \tag{16}$$

we can deduce

$$\lambda = \Phi^\dagger D_\mu D^\mu \Phi = -D_\mu \Phi^\dagger D^\mu \Phi, \tag{17}$$

and the field equation becomes

$$D_\mu D^\mu \Phi + \Phi(D_\mu \Phi^\dagger D^\mu \Phi) = 0. \tag{18}$$

3 Energy Bound

As in the commutative case, the $CP(n)$ model on the noncommutative plane has the Bogomolny energy bound. The conserved energy functional becomes

$$\begin{aligned} E &= \mathrm{Tr}\left(D_0 \Phi^\dagger D_0 \Phi + D_i \Phi^\dagger D_i \Phi) \right) \\ &= \mathrm{Tr}\left(|D_0 \Phi|^2 + |D_z \Phi|^2 + |D_{\bar{z}} \Phi|^2 \right). \end{aligned} \tag{19}$$

Similar to the commutative case, let us consider an inequality

$$\mathrm{Tr}\left\{ (D_i \Phi \pm i\epsilon_{ij} D_j \Phi)^\dagger (D_i \Phi \pm i\epsilon_{ij} D_j \Phi) \right\} \geq 0. \tag{20}$$

Expending this we obtain

$$\mathrm{Tr}(D_i \Phi^\dagger D_i \Phi) \geq \mp i\epsilon_{ij} \mathrm{Tr}(D_i \Phi^\dagger D_j \Phi). \tag{21}$$

The BPS bound on the energy is then

$$E \geq \mathrm{Tr}(D_0 \Phi^\dagger D_0 \Phi) + 2\pi |Q|, \tag{22}$$

where the $U(1)$ gauge invariant 'topological charge' is

$$Q = -\frac{i}{2\pi} \epsilon_{ij} \mathrm{Tr} D_i \Phi^\dagger D_j \Phi = \frac{\mathrm{Tr} F_{12}}{2\pi}. \tag{23}$$

Contrast to the commutative case, there exists no conserved topological current. Instead, the current

$$J^\mu = \frac{1}{4\pi}\epsilon^{\mu\nu\rho}F_{\nu\rho} \tag{24}$$

is covariantly conserved, $D_\mu J^\mu = 0$. However, this implies that for the localized configurations, the topological charge $Q = \mathrm{Tr}J^0$ is conserved. In the complex coordinate,

$$Q = \frac{1}{2\pi}\mathrm{Tr}\left(|D_z\Phi|^2 - |D_{\bar{z}}\Phi|^2\right). \tag{25}$$

The energy bound is saturated by the configuration which is static in time and satisfies the (anti-)self-dual equation [20], $D_i\Phi \pm i\epsilon_{ij}D_j\Phi = 0$, or in the complex notation,

$$D_{\bar{z}}\Phi = 0 \quad (\text{for self-dual case } Q > 0), \tag{26}$$

$$D_z\Phi = 0 \quad (\text{for anti-self-dual case } Q < 0). \tag{27}$$

4 (Anti-)Self-dual Solitons

To find the (anti-)self-dual configurations, let us try to parameterize the field as follows

$$\Phi = W/\sqrt{W^\dagger W}, \tag{28}$$

where W is an $(n+1)$-dimensional vector. Since $\Phi^\dagger\Phi = 1$, locally one can choose finite W such that $\sqrt{W^\dagger W}(x)$ is invertible. We also introduce an $(n+1)$-dimensional projection operator

$$P = 1 - W\frac{1}{W^\dagger W}W^\dagger, \tag{29}$$

whose kernel is one-dimensional space generated by W vector. In terms of this field variable, the Lagrangian (7) becomes

$$L = \mathrm{Tr}\left(\frac{1}{\sqrt{W^\dagger W}}\partial_\mu W^\dagger P \partial^\mu W \frac{1}{\sqrt{W^\dagger W}}\right), \tag{30}$$

and the topological number is

$$Q = \frac{1}{2\pi}\mathrm{Tr}\left\{\frac{1}{\sqrt{W^\dagger W}}\left(\partial_{\bar{z}}W^\dagger P\partial_z W - \partial_z W^\dagger P \partial_{\bar{z}}W\right)\frac{1}{\sqrt{W^\dagger W}}\right\}. \tag{31}$$

The operator in the trace can be regarded as the topological density operator on the noncommutative space. In terms of W variable, the above Lagrangian in the commutative case has a local scaling symmetry, $W \to W\Delta(x)$ with an arbitrary scalar $\Delta(x)$. What is remarkable about the noncommutative space case is that this scaling symmetry on the Lagrangian still holds. The

projection operator P is independent of Δ and so the Lagrangian and the topological number are still invariant. However, the multicomponent field Φ is not invariant under the local scaling on noncommutative case, contrast to the commutative case. As far as the classical field theory is concerned, we could choose the primary field to be W instead of Φ, and regard the classical theory is invariant under the local scaling. This would be crucial in finding the most general solution.

The self-dual equation then becomes

$$D_{\bar{z}}\Phi = P(\partial_{\bar{z}}W)(W^\dagger W)^{-1/2} = 0, \tag{32}$$

which is equivalent to $\partial_{\bar{z}}W = WV$ for arbitrary scalar V both for commutative and noncommutative cases. For either cases the most general solution is

$$W = W_0(z)\Delta(z, \bar{z}) \tag{33}$$

with $(n+1)$-dimensional holomorphic vector $W_0(z)$ and arbitrary scalar function $\Delta(z, \bar{z})$. As we just argued in the previous paragraph, this arbitrariness is a local scaling and can be scaled away.

Let us consider the self-dual solutions in commutative case, which is well studied before. We choose the scaling so that the $(n+1)$th component of W is chosen to be unity. Then, we get the standard self-dual equation for the n-dimensional vector w such that $W = (w, 1)$

$$\partial_{\bar{z}}w = 0 \quad \text{(for self-dual)}, \tag{34}$$

$$\partial_z w = 0 \quad \text{(for anti-self-dual)}. \tag{35}$$

The most general solution of the above self-dual equation should be a n-dimensional vector whose components are holomorphic functions. These solutions are characterized by its topological charge k: the self-dual solutions carry positive integer charges and the anti-self-dual solutions do negative integers. The general self-dual solutions in commutative case are given in the meromorphic form,

$$w = \frac{1}{P_{n+1}(z)}(P_1(z), P_2(z), \cdots, P_n(z)), \tag{36}$$

where $P_i(z)$ are kth order polynomial of z. The meromorphic function is not holomorphic at poles as

$$\partial_{\bar{z}}\frac{1}{z} = 4\pi\delta^2(z). \tag{37}$$

However, w blows up at poles and so the self-dual equation (32) still holds, making the solutions (36) the most general one.

For the commutative case, the solution (32) is equivalent to the smooth solution

$$\Phi = (P_1(z), P_2(z), \cdots, P_{n+1}(z)) \frac{1}{1 + \sum_i P_i^\dagger P_i} \tag{38}$$

by a singular $U(1)$ gauge transformation $|P_{n+1}|/P_{n+1}(z)$. In this case the vector

$$W = (P_1(z), P_2(z), \cdots, P_{n+1}(z)) \tag{39}$$

has components which are kth order polynormials of z only. This solution has $2(n+1)k + 2n$ real parameters, among which $2n$ are the vacuum moduli parameter for $CP(n)$ space and the rest of which $2(n+1)k$ parameters account for the size and scale parameters of k solitons. Note that the W vector in (39) is holomorphic everywhere.

When we goes to the noncommutative case, we should be more careful. As $z^{-1} = (\bar{z}z)^{-1}\bar{z} = \bar{z}(\bar{z}z + \theta)^{-1}$, we get

$$zz^{-1} = I, \quad z^{-1}z = I - |0><0|. \tag{40}$$

Since $\partial_{\bar{z}} f(z,\bar{z}) = \theta^{-1}[z, f(z,\bar{z})]$,

$$\partial_{\bar{z}} z^{-1} = \theta^{-1}|0><0|, \tag{41}$$

and the solution of type $1/z$ is not holomorphic on noncommutative space. This has the analogue of (37) on noncommutative space.

In addition, we will see later that the solution $1/z$ will have fractional topological charge. On the commutative case, two types of solutions (36) and (38) are gauge equivalent, but that is not true in general on the noncommutative case. While z^{-1} is nonholomorphic, the solutions given in Eq. (38) are polynomial so holomorphic, and so they are solutions of the self-dual equation. This is the most general solution even in the noncommutative space, modulo the local scaling we considered before. Not only they satisfy the self-dual equation (32), these solutions also carry the integer topological numbers.

Let us start with a simple solution in the $CP(1)$ model,

$$W = (az, 1), \tag{42}$$

and so

$$\partial_{\bar{z}} W^\dagger P \, \partial_z W = \frac{a^2}{1 + a^2(\bar{z}z + \theta)}, \tag{43}$$

where we have used $zf(\bar{z}z) = f(\bar{z}z+\theta)z$ and $\bar{z}f(\bar{z}z) = f(\bar{z}z-\theta)\bar{z}$. Thus the topological charge is

$$\begin{aligned} Q_s &= \frac{1}{2\pi}\mathrm{Tr}\left\{\frac{1}{1+a^2\bar{z}z}\left(\partial_{\bar{z}}W^\dagger\, P\, \partial_z W\right)\right\} \\ &= \frac{1}{2\pi}\mathrm{Tr}\left\{\frac{a^2}{(1+a^2\bar{z}z)\left(1+a^2(\bar{z}z+\theta)\right)}\right\}. \end{aligned} \tag{44}$$

With the dimensionless parameter $s = a^2\theta$, the trace becomes

$$\begin{aligned} Q_s &= \frac{1}{2\pi}(2\pi\theta)\sum_{n=0}^{\infty}\frac{a^2}{(1+a^2\theta n)(1+a^2\theta(n+1))} \\ &= s\sum\frac{1}{(1+sn)(1+s(n+1))} \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{1+sn}-\frac{1}{1+s(n+1)}\right) = 1. \end{aligned} \tag{45}$$

The scale parameter a of the soliton can be arbitrary but the topological number does not change. Especially for the zero size soliton $a=\infty$, topological charge density does not vanish only for the $|0>$ state.

If we have used the unacceptable singular solution

$$W = (z^{-1}, 1), \tag{46}$$

then its topological charge becomes

$$\begin{aligned} Q_s &= \sum_{n=0}^{\infty}\{1/(1+s(n+1)) - 1/(1+s(n+2))\} \\ &= 1/(1+s). \end{aligned} \tag{47}$$

As we argued before, this solution is not acceptable. Notice that one can see that this solution has the topological charge less than 1. This fractional topological number contrasts with the commutative case. As on the noncommutative case z^{-1} is as singular as z in the operator sense, one see that there can be a configuration with a fractional topological charge. One thus wonder whether one should include this configuration in the family of classically acceptable configurations. We do not know the answer for this. This may be answerable by considering what kind of soliton and anti-soliton pairs are created when some amount of energy is put to the vacuum.

For a single anti-soliton solution in $CP(1)$ model,

$$W = (a\bar{z}, 1), \tag{48}$$

we get

$$\partial_z W^\dagger \, P \, \partial_{\bar z} W = \frac{a^2}{1 + a^2 \bar z z} \quad . \tag{49}$$

Its topological charge is then

$$\begin{aligned} Q_{\bar s} &= -\frac{1}{2\pi} \mathrm{Tr} \left\{ \frac{a^2}{(1 + a^2(\bar z z + \theta))(1 + a^2 \bar z z)} \right\} \\ &= -s \sum_{n=0}^{\infty} \frac{1}{(1 + sn)(1 + s(n+1))} = -1 \quad , \end{aligned} \tag{50}$$

with $s = a^2\theta$.

Thus the topological index works fine for these solitons (38). This suggests that the topological charge can be calculated for arbitrary (anti-)self-dual solutions. This is indeed true as we will see now. For the general solution of Eq. (39), we can say

$$W = a z^k u + \mathcal{O}(z^{k-1}), \tag{51}$$

where u is a $n+1$ dimensional unit vector. To calculate its topological charge, we first note that Eq.(31) can be rewritten as

$$Q_s = \frac{1}{2\pi} \mathrm{Tr} \left\{ \sqrt{W^\dagger W} \partial_{\bar z} \left(\frac{1}{W^\dagger W} W^\dagger \partial_z W \right) \frac{1}{\sqrt{W^\dagger W}} \right\} . \tag{52}$$

Now we can insert the complete set of states between operators to get

$$\begin{aligned} Q_s &= \theta \sum_{n,m,l} < n | \sqrt{W^\dagger W} | m >< m | \partial_{\bar z} \left\{ \frac{1}{W^\dagger W} \partial_z (W^\dagger W) \right\} | l >< l | \frac{1}{\sqrt{W^\dagger W}} | n > \\ &= \theta \sum_{n=0}^{\infty} < n | \partial_{\bar z} \left(\frac{1}{W^\dagger W} W^\dagger \partial_z W \right) | n > \quad . \end{aligned} \tag{53}$$

This is the analogue of the total derivative on noncommutative plane. Noting $\theta \partial_{\bar z} \mathcal{O}(z, \bar z) = [z, \mathcal{O}]$, we can find the integration of the total derivative as [17]

$$\begin{aligned} \frac{1}{2\pi} \mathrm{Tr} \partial_{\bar z} \mathcal{O} &= \sum_{n=0}^{\infty} < n | [z, \mathcal{O}] | n > \\ &= \sum_{n=0}^{\infty} \left\{ \sqrt{\theta(n+1)} < n+1 | \mathcal{O} | n > - \sqrt{\theta n} < n | \mathcal{O} | n-1 > \right\} \\ &= \lim_{N \to \infty} \sum_{n=0}^{N} \{ < n+1 | \mathcal{O} z | n+1 > - < n | \mathcal{O} z | n > \} \\ &= \lim_{N \to \infty} < N+1 | \mathcal{O} z | N+1 > , \end{aligned} \tag{54}$$

assuming that $\mathcal{O} z |0 >= 0$. If $\mathcal{O}$ is singular and so that $\mathcal{O} z |0 > \neq 0$, there would be additional boundary terms. (For example, the singular $w = u/(az^k)$ solution there is an additional boundary term.) If the large N limit vanishes, say like $1/N$, then there is no boundary term and so

the sum vanishes. If the limit is of order one, the limit is finite. If it diverges like a power of N, then the limit is not well defined. In this case, the series should be treated more carefully.

Using the method in (54), we get

$$Q_s = \sum_{n=0}^{\infty} \left\{ < n+1| \frac{1}{W^\dagger W} W^\dagger (\partial_z W) z |n+1> - < n| \frac{1}{W^\dagger W} W^\dagger (\partial_z W) z |n> \right\}. \tag{55}$$

The expectation $< N+1|(W^\dagger W)^{-1} W^\dagger (\partial_z W) z |N+1>$ is of order one. More concretely, we see that

$$\frac{1}{W^\dagger W} W^\dagger (\partial_z W) z = \frac{1}{W^\dagger W} (W^\dagger (\partial_z W) z - k W^\dagger W) + k. \tag{56}$$

Defining that

$$\Omega \equiv \frac{1}{W^\dagger W} \left(W^\dagger (\partial_z W) z - k W^\dagger W \right) , \tag{57}$$

we see

$$\lim_{N \to \infty} < N+1|\Omega|N+1 >= \lim_{N \to \infty} \frac{N^{k-1}}{N^k} = 0 , \tag{58}$$

as $W^\dagger (\partial_z W) z - k W^\dagger W = \mathcal{O}((\bar{z} z)^{k-1})$. Thus the charge becomes

$$\begin{aligned} Q_s &= \lim_{N \to \infty} \{ < N+1|\Omega|N+1 > +k \} \\ &= k. \end{aligned} \tag{59}$$

For general anti-self-dual soliton solution,

$$W = a \bar{z}^k u + \mathcal{O}(\bar{z}^{k-1}), \tag{60}$$

similar argument leads to the topological charge $-k$.

5 Concluding Remarks

In this Letter we have shown that the $CP(n)$ model can be also well-defined on noncommutative two plane. There exist the (anti-)self-dual solitons that saturate the BPS energy bounds, which are regular and carry integer topological charge k with $2(n+1)k+2n$ real parameters. We found that (anti-)self-dual solitons carry integer topological charge regardless their orientation and size when the field configurations are regular. We have also shown that the singular solutions, which are acceptable and related to the regular solutions by gauge transformations on the commutative plane, are not acceptable on noncommutative plane. Not only they do not satisfy the self-dual equations, but also are not related to the regular solution by the gauge transformation on the

noncommutative plane. As we have seen, the topological number does not change when the soliton shrinks to zero size. This should remain true after going to the commutative variables by using the Seiberg-Witten map [4]. While it is not clear how that is achieved in our case, there is no natural way to evoke something like freckled instantons and make two space to blow up at some points, contrast to Braden and Nekrasov's work [12].

The low energy dynamics of the solitons will be described on the moduli space. To do this, one has to know the metric of the moduli space [23]. Our general solutions for k solitons has $2(n+1)k+2n$ real parameters in $CP(n)$ model. The vacuum moduli space has $2n$ real parameters and their kinetic energy diverges due to the volume factor. In addition, the total scale and orientations with $2n$ real parameters have infinite inertia. So a single soliton with $k=1$ has only two parameters with finite inertia, corresponding to the position of the soliton. For k solitons, the moduli space $\mathcal{M}_{k,n}$ of finite inertia has $2(n+1)k-2n$ real dimension. The low energy dynamics of these k solitons can be described by the metric of the moduli space. The solitons would not feel the noncommutativity of the underlying space directly. However, the moduli space of solitons on noncommutative space would be no longer singular when a single soliton collapses to a point. It would be interesting to study in detail the moduli space dynamics of solitons and compare that with those on the commmutative plane.

The solitons in the gauge theories are more complicated than the scalar theory like $CP(n)$. Recently there has been many works on the instantons on the noncommutative $\mathbf{R}^4$ [9, 10, 11, 12, 13, 14]. The soliton properties of the noncommutative $CP(n)$ model may play an important role in figuring out the subtle issues in the noncommutative instantons of four dimensional Yang-Mills theory. One of the key observation from our work is that the topological number on the noncommutative space is somewhat tricky quantity, which needs a careful treatment. Clearly one see a possible solution to the recently discovered quandary [14] where the instanton number of a single $U(2)$ instanton depends on the size of the instanton.

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